Upwards homogeneity of symmetric extensions

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1st February 2024

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Definition

For a notion of forcing $\mathbb{P}$, a $\mathbb{P}$**-name** $\dot{x}$ is a set of tuples $\langle p, \dot{y} \rangle$, where $p \in \mathbb{P}$ and $\dot{y}$ is a $\mathbb{P}$-name. We say that $\dot{y}$ **appears in** $\dot{x}$ whenever there is p such that $\langle p, \dot{y} \rangle \in \dot{x}$.

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Lemma (Symmetry Lemma)

For all formulae $\varphi(\dot{x})$ and $\pi \in \mathrm{Aut}(\mathbb{P})$,

$$p \Vdash \varphi(\dot{x}) \quad \textit{if and only if} \quad \pi p \Vdash \varphi(\pi\dot{x})$$

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- $\mathscr{G} \in \mathscr{F}$;
- if $H \leqslant H'$ and $H \in \mathscr{F}$ then $H' \in \mathscr{F}$; and
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We define hereditary symmetry recursively: If $\dot{x}$ is $\mathscr{F}$-symmetric and, for all $\dot{y}$ appearing in $\dot{x}$, $\dot{y}$ is hereditarily $\mathscr{F}$-symmetric, we say that $\dot{x}$ is **hereditarily** $\mathscr{F}$**-symmetric**.

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A **symmetric system** is a triple $\mathscr{S} = \langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ such that:

- $\mathbb{P}$ is a notion of forcing;
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Theorem (Scott, Jech (?) See [1])

If $V \vDash \mathsf{ZF}$ and $G \subseteq V$ is $\mathbb{P}$-generic, then

$$V[G]_{\mathscr{S}} := \left\{ \dot{x}^G \mid \dot{x} \in \mathsf{HS}_{\mathscr{F}} \right\}$$

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You will more commonly see $\mathsf{HS}_{\mathscr{F}}^G$ used for $V[G]_{\mathscr{S}}$.

Cohen's first model

- $\mathbb{P} = \mathrm{Add}(\omega, \omega)$: Conditions are partial functions $p \colon \omega \times \omega \to 2$ such that $\mathrm{dom}(p)$ is finite and $q \leqslant p$ if and only if $q \supseteq p$.

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  For $n < \omega$, let $\dot{a}_n = \{\langle p, \check{m} \rangle \mid p(n,m) = 1\}$. Let $\dot{A} = \{\langle \mathbb{1}, \dot{a}_n \rangle \mid n < \omega\}$.

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  Therefore $\mathrm{fix}(\{n\}) \leqslant \mathrm{sym}(\dot{a}_n)$ and $\mathscr{G} = \mathrm{sym}(\dot{A})$, so $\dot{a}_n$ and $\dot{A}$ are hereditarily symmetric.

Example: $\pi = (0 \;\; 4 \;\; 2)$

ω

$\vdots$						$\iddots$
1	0		0			
	1	0			0	
0		1	1		0	$\cdots$
0	1	2	3	4	5	

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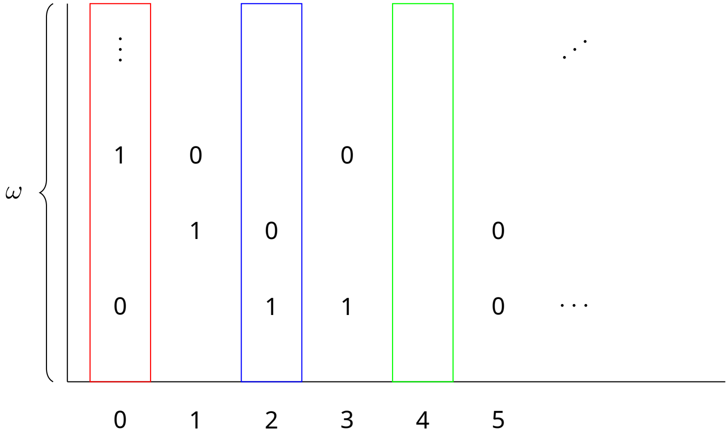

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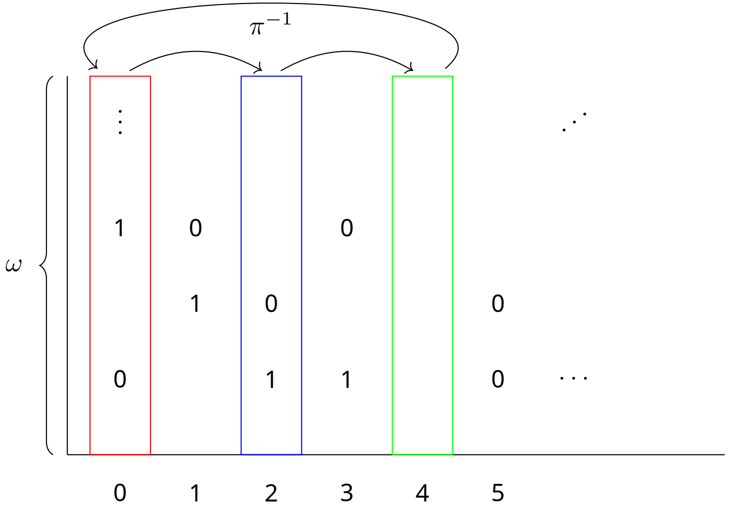

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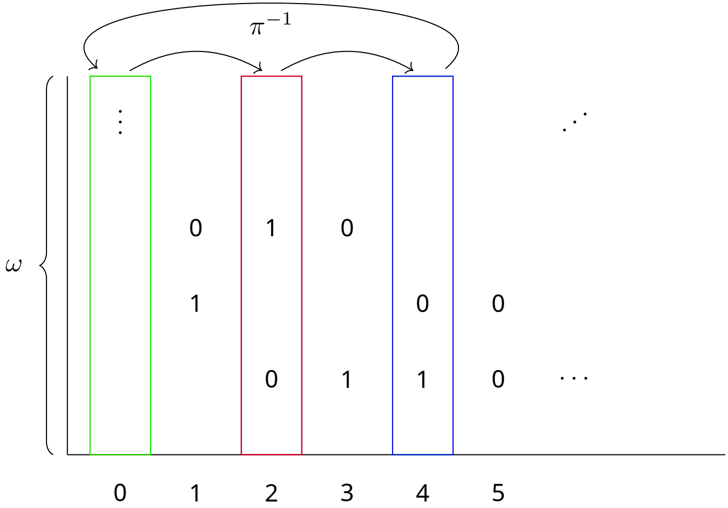

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Proof.

Let $\dot{f} \in \mathrm{HS}$ be a name for a function $\check{\omega} \to \dot{A}$. Since $\dot{f} \in \mathrm{HS}$, there is $E \in [\omega]^{<\omega}$ such that $\mathrm{fix}(E) \leqslant \mathrm{sym}(\dot{f})$.

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$\operatorname{dom}(p)$ is finite, so there is $k \notin E$ such that for all $l < \omega$, $\langle k, l \rangle \notin \operatorname{dom}(p)$. Let $\pi = (k \quad n)$.

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Then $\pi p \Vdash \pi \dot{f}(\pi \check{m}) = \pi \dot{a}_n$. However, $\pi \dot{f} = \dot{f}$, $\pi \check{m} = \check{m}$, $\pi \dot{a}_n = \dot{a}_k$, and $\pi p \parallel p$. Therefore

$$\pi p \cup p \Vdash \dot{f}(\check{m}) = \dot{a}_n \wedge \dot{f}(\check{m}) = \dot{a}_k \quad (\dot{f} \text{ is not a funtion}).$$

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$\mathscr{S}$ is **upwards homogeneous** if for all $H_0 * \dot{H}_1 \in \mathscr{F}$ there is a dense set of conditions $\langle p^\circ, \dot{q}^\circ \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ that are $H_0 * \dot{H}_1$**-controlling**. That is:

For all $\langle p, \dot{q} \rangle, \langle p, \dot{q}' \rangle \leqslant \langle p^\circ, \dot{q}^\circ \rangle$ there is $\bar{\pi} \in H_0 * \dot{H}_1$ such that $\bar{\pi}\langle p, \dot{q} \rangle \parallel \langle p, \dot{q}' \rangle$.

Theorem (RS.–Schilhan–Wei)

An iteration of symmetric systems $\mathscr{S} = \mathscr{S}_0 * \dot{\mathscr{S}}_1$ is upwards homogeneous if and only if for all V-generic $G \times H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$,

$$\mathscr{P}(V) \cap V[G]_{\mathscr{S}_0} = \mathscr{P}(V) \cap V[G][H]_{\mathscr{S}}$$

In particular, there are no new sets of ordinals in $V[G][H]_{\mathscr{S}}$ that did not already appear in $V[G]_{\mathscr{S}_0}$.

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Sketch proof of $\implies$ direction.

We aim to show that if $\dot{X}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a subset of V with $H_0 * \dot{H}_1 \leqslant \mathrm{sym}(\dot{X})$ and $\langle p^\circ, \dot{q}^\circ \rangle$ is $H_0 * \dot{H}_1$-controlling, then whenever $\langle p, \dot{q} \rangle \leqslant \langle p^\circ, \dot{q}^\circ \rangle$ is such that $\langle p, \dot{q} \rangle \Vdash \check{x} \in \dot{X}$, we have $\langle p, \dot{q}^\circ \rangle \Vdash \check{x} \in \dot{X}$.

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Continuation of the proof.

Suppose that $\langle p, \dot{q}^\circ \rangle \nVdash \check{x} \in \dot{X}$. Then there is $\langle p', \dot{q}' \rangle \leqslant \langle p, \dot{q}^\circ \rangle$ such that $\langle p', \dot{q}' \rangle \Vdash \check{x} \notin \dot{X}$.

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We aim to show that if $\dot{X}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a subset of V, with $H_0 * \dot{H}_1 \leqslant \mathrm{sym}(\dot{X})$, then if $\langle p^\circ, \dot{q}^\circ \rangle$ is $H_0 * \dot{H}_1$-controlling, $\langle p, \dot{q} \rangle \leqslant \langle p^\circ, \dot{q}^\circ \rangle$, and $\langle p, \dot{q} \rangle \Vdash \check{x} \in \dot{X}$, then $\langle p, \dot{q}^\circ \rangle \Vdash \check{x} \in \dot{X}$.

If this is the case, then p° forces that $\dot{q}^\circ$ already knows everything about $\dot{X}$, so $\dot{X}^{G \times H} \in V[G]_{\mathscr{S}_0}$.

Continuation of the proof.

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$\langle p', \dot{q} \rangle \leqslant \langle p, \dot{q} \rangle$, so $\langle p', \dot{q} \rangle \Vdash \check{x} \in \dot{X}$. By symmetry,

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This contradicts that $\bar{\pi}\langle p', \dot{q} \rangle \parallel \langle p', \dot{q}' \rangle$. □

Let M be Cohen's first model and let $A \in M$ be the canonical Dedekind-finite set.

For a cardinal κ, let $\mathrm{Col}(A, \kappa)$ be finite partial functions $p \colon A \to \kappa$, with $q \leqslant p$ if and only if $q \supseteq p$.

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Theorem (Karagila–Schlict, [2])

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If H is M-generic for $\mathrm{Add}(A, \kappa)$ then M and $M[H]$ have the same sets of ordinals. □

It is a fact that in M there is no surjection $A \to \omega_1$. However, after forcing with $\mathrm{Col}(A, \kappa)$ we introduce a generic surjection $A \to \kappa$.

Since there are no new sets of ordinals, no cardinals are collapsed.

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- Can other new sets be avoided by similar techniques? E.g., what if we wish to not add any new sets of a certain von-Neumann rank?
- When are upwards homogeneous iterations equivalent to normal forcing iterations?

[1] Thomas Jech. *Set theory*. millennium. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003, pp. xiv+769. ISBN: 3-540-44085-2.

[2] Asaf Karagila and Philipp Schlicht. "How to have more things by forgetting how to count them". In: *Proc. A.* 476.2239 (2020), pp. 20190782, 12. ISSN: 1364-5021,1471-2946. DOI: 10.1098/rspa.2019.0782. URL: https://doi.org/10.1098/rspa.2019.0782.

Thank you